Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ $\qquad$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \;\Rightarrow\; \lim_{x\to a} f(x) = L$

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\;\lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathrm{e}^x = \infty$ & $\lim_{x\to-\infty} \mathrm{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a x^n + \cdots + bx + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a x^n + \cdots + bx + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a x^n + \cdots + cx + d = -\operatorname{sgn}(a)\infty$

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Derivatives

Definition and Notation

If $y=f(x)$ then the derivative is defined to be $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

If $y=f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}(f(x))=Df(x)$$

If $y=f(x)$ all of the following are equivalent notations for derivative evaluated at $x=a$.

$$f'(a)=y'\big|_{x=a}=\frac{df}{dx}\bigg|_{x=a}=\frac{dy}{dx}\bigg|_{x=a}=Df(a)$$

Interpretation of the Derivative

If $y=f(x)$ then,

1. $m=f'(a)$ is the slope of the tangent line to $y=f(x)$ at $x=a$ and the equation of the tangent line at $x=a$ is given by $y=f(a)+f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x=a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)'=cf'(x)$
2. $(f\pm g)'=f'(x)\pm g'(x)$
3. $(fg)'=f'g+fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c)=0$
6. $\frac{d}{dx}(x^n)=nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x)))=f'(g(x))g'(x)$
   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x)=1$

$\frac{d}{dx}(\sin x)=\cos x$

$\frac{d}{dx}(\cos x)=-\sin x$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\frac{d}{dx}(\sec x)=\sec x\tan x$

$\frac{d}{dx}(\csc x)=-\csc x\cot x$

$\frac{d}{dx}(\cot x)=-\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$

$\frac{d}{dx}(a^x)=a^x\ln(a)$

$\frac{d}{dx}(\mathbf{e}^x)=\mathbf{e}^x$

$\frac{d}{dx}(\ln(x))=\frac{1}{x},\ x>0$

$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\ x\neq 0$

$\frac{d}{dx}(\log_a(x))=\frac{1}{x\ln a},\ x>0$

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Extrema

Absolute Extrema

1. $x = c$ is an absolute maximum of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain.
2. $x = c$ is an absolute minimum of $f(x)$ if $f(c) \leq f(x)$ for all x in the domain.

Fermat's Theorem

If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$.

Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ then there exist numbers c and d so that, **1.** $a \leq c, d \leq b$, **2.** $f(c)$ is the abs. max. in $[a,b]$, **3.** $f(d)$ is the abs. min. in $[a,b]$.

Finding Absolute Extrema

To find the absolute extrema of the continuous function $f(x)$ on the interval $[a,b]$ use the following process.

1. Find all critical points of $f(x)$ in $[a,b]$.
2. Evaluate $f(x)$ at all points found in Step 1.
3. Evaluate $f(a)$ and $f(b)$.
4. Identify the abs. max. (largest function value) and the abs. min.(smallest function value) from the evaluations in Steps 2 & 3.

Relative (local) Extrema

1. $x = c$ is a relative (or local) maximum of $f(x)$ if $f(c) \geq f(x)$ for all x near c.
2. $x = c$ is a relative (or local) minimum of $f(x)$ if $f(c) \leq f(x)$ for all x near c.

1st Derivative Test

If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. a rel. max. of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$.
2. a rel. min. of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$.
3. not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$.

2nd Derivative Test

If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. is a relative maximum of $f(x)$ if $f''(c) < 0$.
2. is a relative minimum of $f(x)$ if $f''(c) > 0$.
3. may be a relative maximum, relative minimum, or neither if $f''(c) = 0$.

Finding Relative Extrema and/or Classify Critical Points

1. Find all critical points of $f(x)$.
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) then there is a number $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Newton's Method

If x_n is the n^{th} guess for the root/solution of $f(x) = 0$ then $(n+1)^{st}$ guess is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ provided $f'(x_n)$ exists.

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Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x) = \int f(x)\,dx$) then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\,f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x)\,f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x)\,f[u(x)] - v'(x)\,f[v(x)]$$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_b^a g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = kx + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

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Study Guide on Areas and Volumes Test

Area:

Area: $\int_a^b (f(x) - g(x))dx$

(Top function – bottom function) (a and b are x-limits)

Area = $\int_a^b ((f(y) - g(y))\ dy$

(Right function – left function) (a and b are y-limits)

Volumes -Disk method

- Volume: Use it whenever the axis of rotation is part of the region.
- Always find the intersection points of the functions by using algebra or the 'intersect' feature of your calculator.
- $\pi \int_a^b (f(x))^2 dx \quad axis\ of\ rotation = x - axis$
- $\pi \int_a^b (f(y))^2 dx \quad axis\ of\ rotation = y - axis$
- $\pi \int_a^b (a - f(x))^2 dx \quad axis\ of\ rotation\!: y = a\ where\ y = a\ is\ the\ upper\ boundary$
- $\pi \int_a^b (f(x) - a)^2 dx \quad axis\ of\ rotation\!: y = a\ where\ y = a\ is\ the\ lower\ boundary$
- $\pi \int_a^b (a - f(y))^2 dx \quad axis\ of\ rotation\!: x = a\ where\ x = a\ is\ the\ right\ boundary$
- $\pi \int_a^b (f(y) - a)^2 dx \quad axis\ of\ rotation\!: x = a\ where\ x = a\ is\ the\ left\ boundary$

Volumes-Washer method and Shell method

- Volume: Use it whenever the axis of rotation is **NOT** part of the region.
- Always find the intersection points of the functions by using algebra or the 'intersect' feature of your calculator.
- $\pi \int_a^b ((f(x))^2 - (g(x))^2)dx \quad axis\ of\ rotation = x - axis$

- Horizontal line rotations: Remember f(x) is always the function upper function and g(x) is the lower function.
- $\pi \int_a^b ((f(x) - a)^2 - (g(x) - a)^2)dx \quad axis\ of\ rotation\!: y = a\ where\ y = a\ below\ the\ region.$
- $\pi \int_a^b ((a - g(x))^2 - (a - f(x))^2)dx \quad axis\ of\ rotation\!: y = a\ where\ y = a\ above\ the\ region.$

- Vertical line rotations-Shell: Remember f(x) is always the function upper function and g(x) is the lower function
- $y - axis\!: 2\pi \int_a^b x(f - g)dx$
- $Left\!: x = k\ :\ 2\pi \int_a^b (x - k)(f - g)dx$
- $Right\!: x = k\!:\ 2\pi \int_a^b (k - x)(f - g)dx$

VOLUME OF SOLIDS WITH KNOWN CROSS-SECTIONS

If the known cross-sections are perpendicular to x-axis:

V = $\int_a^b A(x)dx = \int_a^b (f(x) - g(x))$ (Top function – bottom function) where f(x) – g(x) is the base of the solid.

If the known cross-sections are perpendicular to y-axis:

V = $\int_a^b A(y)dx = \int_a^b (f(y) - g(y))$ (Right function – left function) where f(y) – g(y) is the base of the solid.

Formulas:

Cross-section type	**Volume**
Square	V = $\int_a^b (f - g)^2 dx$ or dy
Semi-circle	V = $\frac{\pi}{8} \int_a^b (f - g)^2 dx$ or dy
Equilateral triangle	V = $\frac{\sqrt{3}}{4} \int_a^b (f - g)^2 dx$ or dy
Isosceles triangle with given height	V = $\frac{h}{2} \int_a^b (f - g) dx$ or dy
Rectangle: height is n times the base	V = n$\int_a^b (f - g)^2 dx$ or dy
Isosceles right triangle with leg in R	V = $\frac{1}{2} \int_a^b (f - g)^2 dx$ or dy
Isosceles right triangle with hypotenuse in R	V = $\frac{1}{4} \int_a^b (f - g)^2 dx$ or dy